Directional Analysis of Seismic Data

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1 Introduction

We develop a formalism for estimating the modal content of a seismic wave field, using a limited number of seismic measurements.

2 Formalism

Let us start with the shear and pressure wave fields, $\vec{s}(\vec{x},t)$ and $\vec{p}(\vec{x},t)$ (others can be added as well). The plane wave expansions of these are:

$$\vec{s}(\vec{x},t) = \sum_A \int df d\hat{\Omega}\; S_A(f,\hat{\Omega})\; \vec{e}_A(\hat{\Omega})\; e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/v_s)} \tag{1}$$

$$\vec{p}(\vec{x},t) = \int df d\hat{\Omega}\; P(f,\hat{\Omega})\; \hat{\Omega}\; e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/v_p)} \tag{2}$$

Here, $\hat{\Omega}$ denotes the wave propagation direction, f is frequency, $S_A(f,\hat{\Omega})$ is the amplitude of the shear wave of polarization A, defined by the unit vector $\vec{e}_A(\hat{\Omega})$ that is perpendicular to the wave propagation direction $\hat{\Omega}$), $P(f,\hat{\Omega})$ is the amplitude of the pressure wave which has a single polarization in the longitudinal direction ($\hat{\Omega}$), and v_s and v_p are the speeds of the shear and pressure waves respectively. We can then define two-point correlations:

$$\langle S_A^*(f,\hat{\Omega})\; S_{A'}(f',\hat{\Omega}')\rangle = \delta_{AA'}\;\delta(f-f')\;\delta^2(\hat{\Omega},\hat{\Omega}')\; H_{S,A}(f,\hat{\Omega}) \tag{3}$$

$$\langle P^*(f,\hat{\Omega})\; P(f',\hat{\Omega}')\rangle = \delta(f-f')\;\delta^2(\hat{\Omega},\hat{\Omega}')\; H_P(f,\hat{\Omega}) \tag{4}$$

$$\langle S_A^*(f,\hat{\Omega})\; P(f',\hat{\Omega}')\rangle = 0 \tag{5}$$

Here, δ's denote the Kronecker or Dirac delta functions, and H's denote the power in the shear waves of polarization A or in pressure waves. These assumed two-point correlations essentially state that waves at different frequencies, from different directions, and of different polarization (shear or pressure) are all uncorrelated.

Seismometer i at a location $\vec{x}_i$ then measures in the $\hat{x}$ direction:

$$d_{i,x}(\vec{x}_i, t) = (\vec{s}(\vec{x}_i, t) + \vec{p}(\vec{x}_i, t)) \cdot \hat{x} \tag{6}$$

and similarly for directions $\hat{y}$ and $\hat{z}$. In principle we should add the seismometer noise to this measurement, but we will assume that the seismic noise floor is significantly higher than the instrument noise. Now we can compute the cross-correlation between channels of two seismometers located at different locations (we denote the channels by α and β, they take values x, y, z and in general need not be the same):

$$\begin{aligned} \langle Y_{i\alpha,j\beta} \rangle &= \int_{-T/2}^{T/2} dt \; d_{i\alpha}(\vec{x}_i, t) d_{j\beta}(\vec{x}_j, t) & (7) \\ &= \int_{-T/2}^{T/2} dt \int df d\hat{\Omega} \left(\sum_A H_{S,A}(f, \hat{\Omega}) e_{A,\alpha}(\hat{\Omega}) e_{A,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega} \cdot \Delta\vec{x}/v_s} \right. \\ &+ \left. H_P(f, \hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega} \cdot \Delta\vec{x}/v_p} \right) & (8) \end{aligned}$$

where we have defined projections $e_{A,\alpha}(\hat{\Omega}) = \vec{e}_A(\hat{\Omega}) \cdot \hat{\alpha}$ and $\Omega_\alpha = \hat{\Omega} \cdot \hat{\alpha}$, and $\Delta\vec{x} = \vec{x}_i - \vec{x}_j$. The time integral is trivial. Also, we can perform the analysis in a small frequency bin Δf so that the frequency integral is also simple (we add a factor of 2 when switching to integration over frequencies between 0 and $+\infty$):

$$\begin{aligned} \langle Y_{i\alpha,j\beta} \rangle &= 2T\Delta f \int d\hat{\Omega} \left(\sum_A H_{S,A}(\hat{\Omega}) e_{A,\alpha}(\hat{\Omega}) e_{A,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega} \cdot \Delta\vec{x}/v_s} \right. \\ &+ \left. H_P(\hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega} \cdot \Delta\vec{x}/v_p} \right) & (9) \end{aligned}$$

where we have suppressed the frequency dependence of the shear and pressure wave amplitudes. We can then parameterize the amplitudes in terms of whatever basis $\{Q_a(\hat{\Omega})\}$ would be most useful:

$$\begin{aligned} H_{S,A}(\hat{\Omega}) &= \sum_a S_{A,a} \, Q_a(\hat{\Omega}) & (10) \\ H_P(\hat{\Omega}) &= \sum_a P_a \, Q_a(\hat{\Omega}) & (11) \end{aligned}$$

One useful basis may be the spherical harmonics, $Q_{lm}(\hat{\Omega}) = Y_{lm}(\hat{\Omega})$. Another basis choice could be "pixels", i.e. specific propagation directions $Q_{\hat{\Omega}_0}(\hat{\Omega}) = \delta(\hat{\Omega} - \hat{\Omega}_0)$. Either way, we can define

$$\begin{aligned}
\gamma_{S1a} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) e_{1,\alpha}(\hat{\Omega}) e_{1,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \\
\gamma_{S2a} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) e_{2,\alpha}(\hat{\Omega}) e_{2,\beta}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_s} \\
\gamma_{Pa} &= \int d\hat{\Omega}\; Q_a(\hat{\Omega}) \Omega_\alpha \Omega_\beta e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/v_p}
\end{aligned} \tag{12}$$

Note we suppressed the indices $ij\alpha\beta$. We can then write

$$\begin{aligned}
\langle Y_{i\alpha,j\beta} \rangle &= 2T\Delta f (S_{1a}\gamma_{S1a} + S_{2b}\gamma_{S2b} + P_c\gamma_{Pc}) \\
&= S_d \gamma_d
\end{aligned} \tag{13}$$

where the repeated indices are summer over (and the sum over the index d in the last line includes all 3 sums in the previous line). The goal of the analysis is to estimate the coefficients S_{1a}, S_{2b}, P_c (or, equivalently, S_d). We define the likelihood as

$$L \propto \exp\Big(- (Y_i^* - \gamma_{id}^* S_d) N^{-1} (Y_i - \gamma_{id} S_d) \Big) \tag{14}$$

where i now runs over all detector/channel pairs, and d runs over all basis elements. The covariance matrix N in our case is simple - we can assume that all detectors and channels have similar noise floors, constant in time, so N then becomes proportional to the identity matrix and we can ignore it in likelihood maximization. The best estimate is then given by

$$\vec{S} = (\gamma^{T*}\gamma)^{-1} \gamma^* \vec{Y} \tag{15}$$

where in the last line we think of γ as a matrix of elements γ_{id}. So the problem reduces to computing the γ matrix, which can be done once the basis is chosen using Eq. 12. Note that it is straightforward to add other components of the seismic wave field.