University of Minnesota
School of Physics and Astronomy

GRADUATE WRITTEN EXAMINATION

SPRING 2003 - PART I

Thursday, January 16, 2003 - 9:00 A.M. to 12:00 NOON

Part I of this exam consists of 12 problems of equal weight. You will be graded on your 10 best efforts.

This is a closed book examination. You may use a calculator. A list of some physical constants and properties that you may require is included; please take a moment to review its contents before starting the examination.

Please put your **CODE NUMBER** (not your name) in the **UPPER RIGHT-HAND CORNER** of each piece of paper that you submit, along with the relevant problem number in the **UPPER LEFT-HAND CORNER**.

BEGIN EACH PROBLEM ON A FRESH SHEET OF PAPER, so that no sheet contains work for more than one problem.

USE ONLY ONE SIDE of the paper; if you require more than one sheet, be sure to indicate "page 1", "page 2", etc., under the problem number already entered on the sheet.

Once completed, all your work should be put into the manila envelope provided, **IN ORDER** of the problem numbers.

Constants	Symbols	Values
Speed of light in vacuum	c	3.00×10^{8} m/s
Elementary charge	e	1.60×10^{-19} C
Permittivity constant	ϵ_0	8.85×10^{-12} F/m
Permeability constant	μ_0	1.26×10^{-6} H/m
Electron rest mass	m_e	9.11×10^{-31} kg 0.511 MeV/c^2
Proton rest mass	m_p	1.67×10^{-27} kg 0.938 GeV/c^2
Neutron rest mass	m_n	1.68×10^{-27} kg 0.940 GeV/c^2
Planck constant	h	6.63×10^{-34} J.s 4.14×10^{-15} eV.s
Molar gas constant	R	8.31 J/mol.K
Avogadro's number	N_A	6.02×10^{23} /mol
Boltzmann constant	k_B	1.38×10^{-23} J/K 8.62×10^{-5} eV/k
Standard atmosphere		1.01×10^{5} N/m^2
Faraday constant	F	9.65×10^{4} C/mol
Stefan-Boltzmann constant	σ	5.67×10^{-8} W/m^2.K^4
Rydberg constant	R	1.10×10^{7} m^{-1}
Bohr radius	a_0	5.29×10^{-11} m
Gravitational constant	G	6.67×10^{-11} m^3/s^2.kg
Electron magnetic moment	μ_e	9.28×10^{-24} J/T
Proton magnetic moment	μ_p	1.41×10^{-26} J/T
Bohr magneton	μ_B	9.27×10^{-24} J/T
Nuclear magneton	μ_N	5.05×10^{-27} J/T
Earth radius		6.37×10^{6} m
Earth-Sun distance		1.50×10^{11} m
Earth-Moon distance		3.82×10^{8} m
Mass of Earth		5.98×10^{24} kg
Mass of Sun		1.99×10^{30} kg
Mass of Moon		7.36×10^{22} kg

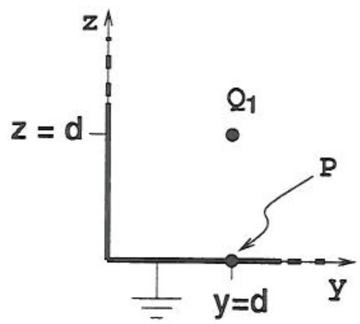
2003 Spring GWE

1. Consider a hydrogen atom located at $\mathbf{r} = 0$. Assume that in addition to the proton Coulomb potential, the electron experiences a small short-range potential $Ua^3\delta(\mathbf{r})$, with $U << 1$ Ry , and a the Bohr radius. Calculate the correction to the energy of the $1s$–$2p$ transition. (This model accounts for the finite size of the nucleus).

2. Experimental measurements of the heat capacity of aluminum at low temperatures (below about 50 K) can be fit to the formula $C_v = aT + bT^3$ where C_v is the heat capacity of one mole of aluminum, with a = 0.00135 J/K^2 and b = 2.48 x 10^{-5} J/K^4. From this data, find a formula for the entropy of a mole of aluminum as a function of temperature. Evaluate your formula at T = 1 K and T = 10 K, expressing your answers in conventional units (J/K).

3. Two arbitrarily shaped conducting objects with electric charge $+Q_0$ and $-Q_0$, respectively, are immersed in a medium with conductivity s. What is the current that flows between the two objects as a function of time?

4. The figure shows a positive point charge Q_1 located at the point *(0, d, d)*. Two thin semi-infinite, grounded, conducting plates lying in the *x-z* and *x-y* planes meet at the *x*-axis as shown.



a) What is the electric field at the point *P* shown, right at the surface of the conductor (*i.e.*, at z=ε, as $\varepsilon \to 0$) ?

b) Determine the local surface charge density at this point *P*.

5. A recent Fermilab experiment obtained direct evidence for the existence of the tau-neutrino ν_τ by observing the reaction $\nu_\tau + n \to p + \tau^-$, where the produced τ^- was tracked in emulsion detectors before it decayed. Assuming the neutron n is at rest, what is the minimum energy that the ν_τ must have so that this reaction can take place? (For this problem, ignore the mass of the ν_τ as well as the proton-neutron mass difference, and use $m_\tau = 1777$ MeV/c^2 for the τ^- mass.)

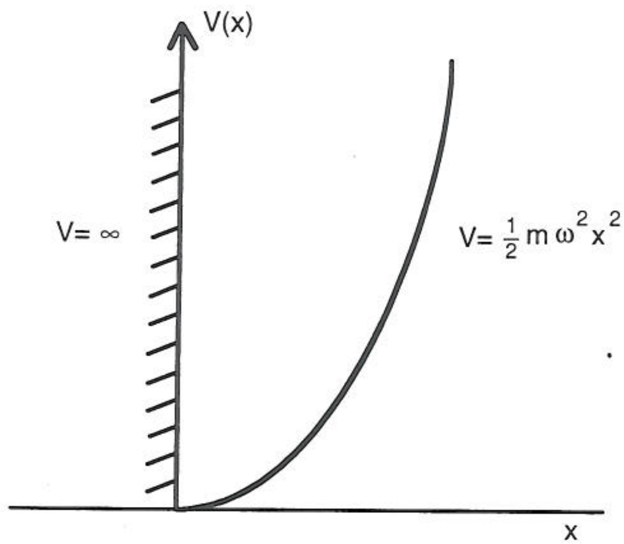
6. NASA's Polar satellite is in an orbit around Earth with an apogee of 9 R_E and a perigee of 1.8 R_E, measured from the center of the Earth (1 R_E = 1 Earth radius = 6380 km). Determine its semi-major axis, eccentricity and orbital period.

7. A particle of mass m slides without friction on the surface of a sphere of radius a. If the particle starts at rest on the top of the sphere, where does the particle leave the surface of the sphere?

8. Evaluate, correct to five decimal places $\int_0^{0.1} \frac{e^x - 1}{x} dx$

9. Find the ground state energy for particle in the following potential

$$V(x) = \tfrac{1}{2} m\omega^2 x^2 \text{ at } x > 0, \qquad V(x) = \infty \text{ at } x \le 0$$



10. For the state $\psi = A\cos^2\varphi$ where φ is the azimuthal angle, determine the probabilities for different m, the z-projection of angular momentum.

11. In the course of pumping up a bicycle tire, a liter of air at 300 K and atmospheric pressure is compressed adiabatically to a pressure of 7 atm. (Air is mostly diatomic nitrogen and oxygen).

a) What is the final volume of the air after compression?
b) How much work is done in compressing the air?
c) What is the temperature after compression?

12. A vertical square loop of copper with sides of length, l, is falling from a region in which the magnetic field is uniform and of strength B into a region in which the magnetic field is zero. At $t = 0$ the lower edge of the square crosses the boundary into the region of zero magnetic field. The radius of the wire is r, the mass density of the copper is ρ_m, and the conductivity is σ. Assume the copper square reaches a steady velocity (while still crossing the boundary), calculate this velocity.

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GRADUATE WRITTEN EXAMINATION

SPRING 2003 - PART 2

Friday, January 17, 2003 - 9:00 A.M. to 1:00 P.M.

Part 2 of this exam consists of 6 problems of equal weight. You will be graded on your 5 best efforts.

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Long problems

1) a) Calculate the transmission coefficient for the delta-function potential $U(x) = \alpha\delta(x\text{-}x_0)$.

Hint: show that the Schrödinger equation with the delta-function potential leads to the following matching condition:

$$\lim_{\varepsilon \to 0} \left[\frac{d\,\psi}{d\,x}(x_0+\varepsilon) - \frac{d\,\psi}{d\,x}(x_0-\varepsilon) \right] = \frac{2m\alpha}{\hbar^2}\psi(x_0)$$

b) Find the equation for the energies E at which a particle is not reflected by a potential consisting of two delta-functions:

$$U(x) = \alpha\,\{\delta(x) + \delta(x\text{-}a)\}.$$

Solve this equation in the limit

$$\alpha >> \sqrt{\frac{\hbar^2 E}{m}}$$

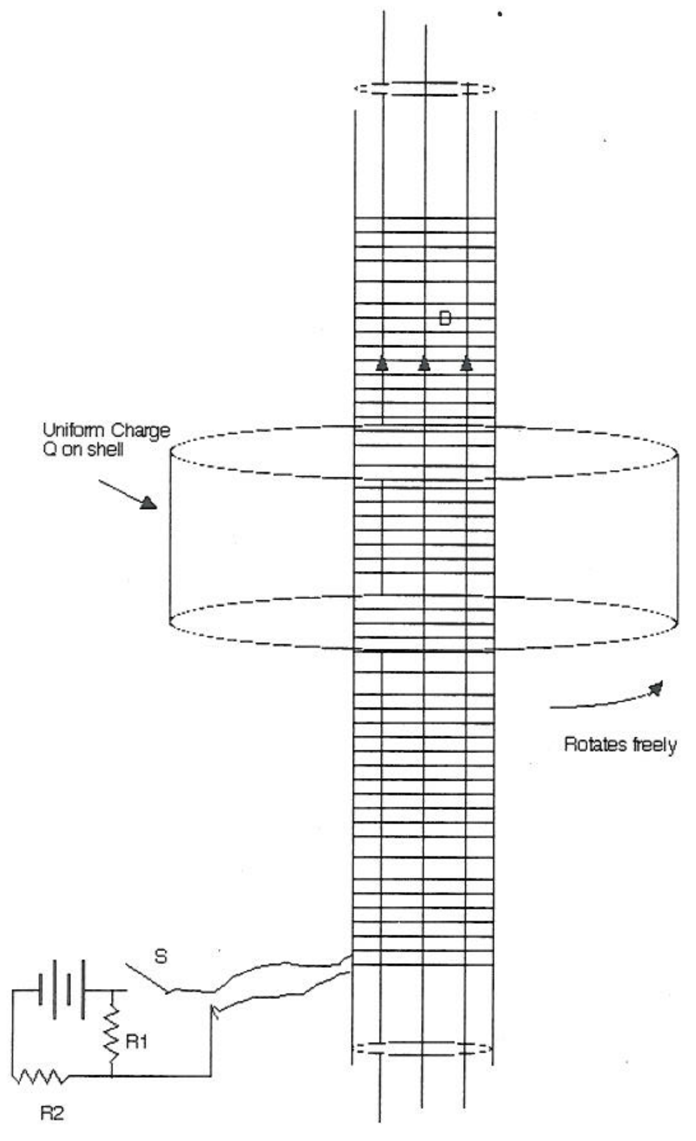
2. Heat capacities are normally positive, but there is an important class of exceptions: systems of particles held together by gravity, such as stars and star clusters.
   a) Consider a system of just 2 particles, with identical masses, orbiting in circles about their center of mass. Show that the gravitational potential energy of the system is -2 times the total kinetic energy.
   b) The conclusion of part a) turns out to be true, at least on average, for any system of particles held together by mutual gravitational attraction: $\overline{U}_{potential} = -2\overline{U}_{kinetic}$ here each $\overline{U}$ refers to the total energy (of that type) for the entire system, averaged over some sufficiently long time. (This result is the Virial theorem). Suppose, then, that you add some energy to such a system and then wait for the system to equilibrate. Does the average total kinetic energy increase or decrease? Explain.
   c) A star can be modeled as a gas of particles that interact with each other only gravitationally. According to the equipartition theorem, the average kinetic energy of the particles in such a star should be $\frac{3}{2}kT$, where T is the average temperature. Express the total energy of the star in terms of its average temperature, and calculate the heat capacity. Note the sign.
   d) Derive the gravitational potential energy for the star of mass M and radius R.
   e) Estimate the average temperature of the sun, whose mass is 2 x 10^{30} kg, and whose radius is 7 x 10^{8} m. Assume, for simplicity, that the sun is made entirely of protons and electrons.

3. The phenomenon of "neutrino oscillations" can occur if (1) the neutrino eigenstates of the weak interaction are distinct from the mass eigenstates and (2) the neutrino masses are non-zero and non-degenerate. Consider the case where two of the neutrino types satisfy these conditions, with $|\nu_e\rangle$ and $|\nu_\mu\rangle$ representing the weak eigenstates, while $|\nu_1\rangle$ and $|\nu_2\rangle$ represent mass eigenstates with masses m_1 and m_2, such that: $|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$ and $|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$. Fusion in the sun produces $|\nu_e\rangle$'s with momentum $p \sim 10$ MeV/c. Assuming that the mass eigenstates propagate as plane-wave states with time-dependence given by $e^{i2\pi Et/h}$, and that m_1 and $m_2 << pc$, and ignoring neutrino interactions inside the sun,

a) What is the value of θ if all of the solar $|\nu_e\rangle$'s with this momentum have 'oscillated' into $|\nu_\mu\rangle$'s when they reach the earth?

b) What is the smallest possible value of $\delta m^2 = |m_1^2 - m_2^2|$ that could account for this observation?

4. A nearly infinite solenoid with N turns / m has radius a, and carries current I. The solenoid is connected to a circuit with a battery, a switch, and resistors R_1 and R_2. A large cylindrical shell with radius b and length L surrounds the solenoid as shown below. The cylinder is free to rotate frictionlessly about its axis. The cylinder is an insulator with charge Q sprayed uniformly over its surface. If the switch is opened and the current from the battery interrupted, what is the angular momentum of the shell after the current has died out?



5. Consider a double pendulum, i.e., a system consisting of 2 simple pendula each of length l, and mass m with one pendulum suspended from the other. Calculate the frequencies of small oscillation for this system, and describe the motion for each of the normal modes. Assume the 2 pendula oscillate in the same plane.

6. This problem follows Bethe's theory of the rate of a thermonuclear reaction.
The rate of the thermonuclear reaction of conversion of two deuterons to helium is very slow because of the Coulomb repulsion $U(r) = e^2/r$ between them. Only at small distance $r_0 \sim 10^{-15}$ m do they reach the very deep well of nuclear attraction potential and merge. There are two ways to overcome the Coulomb barrier. You can do it classically or via tunneling.

a) Exploring classical route, assume that deuteron gas with the concentration N_0 has the equilibrium temperature $T = 10^6$ K, and estimate the concentration of deuterons N with such energy that they can reach the distance r_0 classically. It is too small and can not explain the observed rate of the thermonuclear reaction.

b) Exploring the tunneling route, assume that two deuterons collide with kinetic energy ε in their center of mass system. Using the WKB approximation, estimate the main exponential term of probability P to come to the distance r_0 via tunneling under the barrier. If ε is of the order of the thermal energy this rate is very small at the above mentioned star temperature.

c) In 1937 Bethe realized that a much faster alternative route combines both classical and quantum routes. Deuterons with an energy $k_B T \ll \varepsilon \ll e^2/r_0$ can tunnel to meet each other. In this case, the barrier is more transparent than for thermal deuterons, although the concentration of such deuterons is small. He wrote the rate as the product of number of such deuterons and tunneling probability at this energy. Both terms are exponential so he found an exponential dependence of rate R on some function $f(\varepsilon)$. Then optimizing $f(\varepsilon)$ with respect to ε, he found the optimal $\varepsilon = \varepsilon_0$ and arrived at the famous law for the temperature dependence of thermonuclear reactions. The Bethe law was the first example in physics when a reaction rate exponentially depends not on T^{-1} (Arhenius law) but on T^{-q}, where $q < 1$. Follow Bethe and find q.