Problem 1

The potential energy function of a particle of mass m moving along the x-axis is

$$V(x) = \frac{cx}{x^2 + a^2}$$

where c and a are positive constants. First, sketch V as a function of x and then,

(a) Find the position of stable equilibrium, and the period of small oscillations about this position.

Given that the particle starts from this point with velocity v to the right,

(b) Find the ranges of values of v for which it (1) oscillates, (2) escapes to $-\infty$, and (3) escapes to $+\infty$.

Problem 2

The radiant flux of sunlight on the Earth's surface at midday is approximately 1 kW/m^2.

(a) What is the corresponding (rms) magnitude of the magnetic field strength in Tesla?

(b) What is the radiation pressure on a mirror lying on the ground (in Pa)?

Problem 3

A batch of 1000 components of the same type for use in the DUNE neutrino detector is believed to include 5% which are faulty.

(a) If 5 components are selected at random, what is the probability that no defective component will be chosen?

(b) What is the probability that exactly 2 out of the 5 will be defective?

Problem 4

The Moon's mass and radius are 0.0123 M_E and 0.273 R_E (E = Earth). For Jupiter the corresponding figures are 318 M_E and 11.0 R_E. Find in each case the gravitational acceleration at the surface, and the escape velocity in SI units.

Problem 5

A steady current density **J** gives rise to a time-independent magnetic field $\boldsymbol{B} = 2xy\, a\, \widehat{\boldsymbol{x}} + y^2\, b\, \widehat{\boldsymbol{y}}$, where a and b are constants in appropriate SI units (T/m²).

(a) How are a and b related?

(b) What is the current density **J** ?

Problem 6

The starship Enterprise goes to a planet in a star system far away with a speed of 0.9c, spends 6 months on the planet, and comes back with a speed of 0.95c. The entire trip takes 5 years for the crew.

(a) How far (in light-years) is the planet according to Earth observers?

(b) How long (in years) did it take the crew (that is, by starship time) to get to the planet?

(c) How long (in years) did the entire trip take for the Earth observers?

Problem 7

Consider a simple model in which the planets act like black bodies, re-radiating in equilibrium the energy they receive from the Sun. If in such a model the surface temperature of the Earth is predicted to be 281 K (ignoring atmospheric effects), what is the predicted surface temperature of Mars whose orbit around the Sun has a radius of about 1.52 AU? Recall that 1 AU is the Earth-Sun distance.

Problem 8

The electron in a hydrogen atom is in a state described by the following superposition of normalized energy eigenstates u, with real $A > 0$,

$$\psi(r, \theta, \varphi) = \frac{1}{5}(3u_{100} + Au_{211} - 2u_{21-1} + 3u_{321})$$

where the subscripts represent the quantum numbers $\{n, \ell, m_\ell\}$.

(a) Calculate A such that this wavefunction is normalized.

(b) Find the expectation value of the energy in this state, in terms of the ground state energy of hydrogen E_1.

(c) Find the expectation values of L^2 and L_z in this state.

Problem 9

A typical neutron star has approximately the same mass as the sun but is as dense as a proton (of radius 10^{-15} m). Estimate, in order of magnitude, the radius of a neutron star (in km) and the gravitational binding energy released in its formation (in Joules). How does this compare with the Sun's rest energy (that is, $M_{Sun}c^2$)?

Problem 10

One mole of an ideal monoatomic gas undergoes a process for which the relation between pressure and volume is $p = p_0 + a/V$ where p_0 and a are positive constants. The gas expands from an initial volume V_1 to a final volume V_2. Find

(a) the change in the internal energy ΔU of the gas

(b) the work W done by the gas on its surroundings, and

(c) the amount of heat Q_{in} transferred to the gas by its surroundings.

Problem 1

If the Solar System were immersed in a uniformly dense spherical cloud of dark matter, objects within would experience a net gravitational potential energy from both the Sun and the dark matter of the form

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2$$

(a) Using plane polar coordinates (r, φ), write down the Lagrangian for an object of mass m moving in this potential and derive the resulting equations of motion, identifying all conserved quantities. In particular, write down the equivalent one-dimensional equation of motion for r.

(b) Find the condition under which the object will move in a circular orbit of radius R. You need not solve for R !

(c) Find the frequency ω for small radial oscillations δr of the object in a nearly circular orbit in terms of R and the potential parameters k and b.

Problem 2

Consider the sum $\boldsymbol{E} = \boldsymbol{E_1} + \boldsymbol{E_2}$ of the electric fields of two oppositely traveling electromagnetic waves

$$\boldsymbol{E_1} = \hat{\boldsymbol{x}}\, E_0 \cos(\omega t - kz)\,,\, \boldsymbol{E_2} = \hat{\boldsymbol{x}}\, E_0 \cos(\omega t + kz)$$

(a) Show that this corresponds to a standing wave and obtain the corresponding magnetic field.

(b) What are the energy density and Poynting vector for this standing wave?

(c) Find the time-averaged expressions for the energy density and Poynting vector for this standing wave.

Problem 3

Consider the quantum mechanical problem of two identical non-relativistic particles, each of mass m and spin 1/2, confined to a one-dimensional infinite square well of width L (that is, $V = 0$ for $0 < x < L$ and $V \to \infty$ everywhere else). In the first part of the problem the particles are not mutually interacting.

Note: For a single particle of mass m, the normalized eigenfunctions and corresponding energies are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\, sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

(a) Write down the three lowest energy eigenstates of the system which are also eigenstates of total spin, including both the spatial and spin parts (denoted by $|S, S_z\rangle$) of the wave functions, bearing in mind their symmetry properties. Give the energy of each state and make an energy diagram indicating the energies and degeneracies.

(b) Now introduce an interaction between the particles $V(x_1 - x_2) = \lambda\, \delta(x_1 - x_2)$. Find the resulting shifts in the energies of the states found in part (a) in first order perturbation theory in λ. Indicate how the energy levels are shifted in a diagram as in (a).

Problem 4

Grand-unified theories in particle physics generically predict that the proton will decay. The proposed Hyper-Kamiokande experiment in Japan consists of cylindrical tanks each containing ultra-pure water which are lined with ultra-sensitive photo sensors to detect the Cerenkov light produced by the proton decay products.

(a) The tanks at the Hyper-Kamiokande experiment will contain 260,000 metric tons (1 metric ton = 1000 kg) of water. If the mean proton lifetime is 2×10^{34} years, how many decays would you expect to observe in one year? Assume that the detector is 100% efficient and that protons bound in nuclei and free protons decay at the same rate.

(b) A possible proton decay channel is $p \rightarrow \pi^0 + e^+$, where π^0 is a neutral pion and e^+ is a positron. Calculate the positron energy (in MeV) if the proton decays at rest. How does the positron speed compare to the speed of light in water whose refractive index is n = 1.33? The π^0 mass is $m_\pi = 135$ MeV/c^2 . The positron mass $m_e = 0.511$ MeV/c^2 is so much smaller than either of the proton or pion masses that you may set it to zero for this problem.

(c) The π^0 immediately decays in flight to two photons (in 10^{-16} sec), $\pi^0 \rightarrow \gamma + \gamma$. What are the minimum and maximum photon energies (in MeV) to be expected from a proton decaying at rest? (Hint: What is the corresponding configuration of final photon momenta?)

Problem 5

Consider a one-particle system capable of three states ($\varepsilon_n = n\Delta$, n = 0, 1, 2) in thermal contact with a reservoir at temperature T. Find each of the following quantities in terms of Δ and T and obtain their values in the limits $k_B T/\Delta \rightarrow \infty$ and $k_B T/\Delta \rightarrow 0$:

(a) the internal energy.

(b) the free energy.

(c) the heat capacity.