Problem 1

A useful picture of the Earth's magnetic field is that of a dipole (like a bar magnet) through its center, tilted at an angle of 11° with respect to the Earth's rotational axis. While it varies by location, a reasonable average value for the surface magnitude of the field is $B_S = \bar{B}(R) \cong 25\ \mu\text{T}$, with R the Earth's radius. Use this information to estimate the total energy (in Joules) in the external ($r \geq R$) magnetic field of the Earth.

Problem 2

The wave function of a quantum particle that is free to move in one dimension is given by

$$\psi(x) = \sqrt{\frac{1}{2a}}\ ,\quad |x| \leq a$$

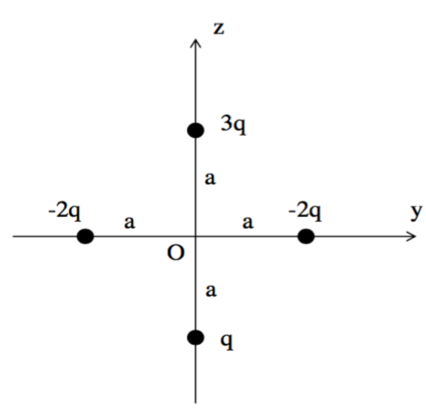
and $\psi(x) = 0$ for $|x| > a$. Find the corresponding momentum probability distribution (normalized to its value at $p = 0$) and show that the probability of finding the particle with momentum $p = \hbar\pi/2a$ relative to the probability of zero momentum is $4/\pi^2$.

Problem 3

Four electric charges are placed as shown in the figure.
As the total charge is zero, there is no monopole moment.

(a) Find the electric dipole moment **p** of this charge distribution and the corresponding leading term for the electrostatic potential, valid far from the origin.

(b) What is the corresponding leading term for the electric field at points (1) $\boldsymbol{r} = z\,\hat{z}$ (on the z-axis) and (2) $\boldsymbol{r} = y\,\hat{y}$ (on the y-axis), valid far from the origin?



Problem 4

A particle of mass m travelling at a speed u collides with a stationary particle of equal mass and they combine to form a new particle.

(a) Calculate the speed V and mass M of the new particle, expressing your result in terms of m, u and $\gamma_u = 1/\sqrt{1 - u^2/c^2}$.

The new particle subsequently breaks up into two particles, each of mass αm.

(b) Obtain the magnitude of the three-momentum of each particle in the rest frame of the decaying particle of mass M and deduce the maximum value of α again in terms of u and γ_u.

Problem 5

The energy levels of the atoms of a certain substance are uniformly separated by an amount $\Delta\epsilon = 3.20 \times 10^{-20}$ J so that they are given by

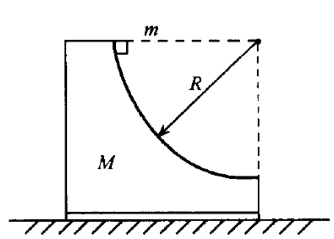
$$\epsilon_n = \epsilon_0 + n\,\Delta\epsilon$$

with integer $n \geq 0$. There is no degeneracy.

What fraction of the atoms are then in the ground state ($n = 0$) when the temperature of the substance is (a) $T = 300$ K, and (b) $T = 1000$ K?

Problem 6

A circular track of mass M and radius R (as shown) is at rest on top of a frictionless surface. The top of the track is perpendicular while its bottom is horizontal. A small block of mass m (whose size you can ignore) is released from rest at the top of the track. When the block leaves the bottom of the track, what are the respective velocities v and V of the block and the track?



Problem 7

In the Stern-Gerlach experiment, silver atoms are passed through an inhomogeneous magnetic field separating those in a spin-up state from those in a spin-down state, and two detectors measure the respective number of atoms in each state. In repeating this experiment, you wish to determine whether your sample of atoms is already partially polarized, with an excess of atoms in one spin-state relative to the other. Unfortunately, your source is not producing many atoms.

You detect 13 atoms in total, of which 12 are in the spin-up state and 1 is in the spin-down state. Assuming there is no net polarization in your sample, what is the probability of observing this final count?

Problem 8

Two moles of a monoatomic ideal gas are at a temperature $T = 300\ K$. The gas expands reversibly and isothermally to twice its original volume. Calculate

(a) The work done by the gas (in J)

(b) The heat supplied to the gas (in J)

(c) The change in the internal energy of the gas (in J)

(d) The change in entropy of the gas (in J/K)

Problem 9

Consider the following time-independent wavefunction

$$\psi(\boldsymbol{r}) = (x + y + 3z)\, f(r)$$

in which $f(r)$ need not be specified further, other than it being a function of $r = \sqrt{x^2 + y^2 + z^2}$ only.

(a) Verify by explicit calculation that $\psi(\boldsymbol{r})$ is not an eigenstate of either one of the three L_x, L_y or L_z.

(b) Show that $\psi(\boldsymbol{r})$ is in fact an eigenstate of L^2 and find the corresponding quantum number ℓ.

(c) Since $\psi(\boldsymbol{r})$ is an eigenstate of L^2, it will also be an eigenstate of $\hat{n} \cdot \boldsymbol{L}$ with eigenvalue $\hbar m$ for some unit vector $\hat{n} = \boldsymbol{n}/|\boldsymbol{n}|$. What are m and $\hat{n}$?

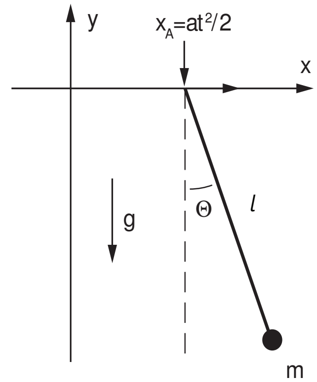
Problem 10

The binary stellar system X consists of two objects orbiting about their common center-of-mass under the influence of their mutual gravitational attraction. One of the objects is a blue supergiant star with a mass $M_{SG} = 25\ M_S$ while the other is a black hole candidate, with a mass $m_{BH} = 10\ M_S$, where M_S is the mass of the Sun. The orbital period of motion is $T_X = 5.6$ days.

Assuming that the orbits of both objects are circular, that is, their relative distance remains constant, make a sketch of this system and from the information given, determine the orbital radii R_{SG} and R_{BH} (in AU) of the supergiant star and black hole, respectively. Note: 1 AU = 1 Astronomical Unit = Earth-Sun distance.

Problem 1

A pendulum consisting of a rigid rod of length ℓ (assumed massless) to which is attached a mass m, performs oscillations in the x-y plane, with y as the vertical direction. In addition, the point of suspension A of the pendulum moves with an (externally determined) constant acceleration a in the x-direction so that its coordinates are $(x_A, y_A) = \left(\frac{at^2}{2}, 0\right)$, as seen in the figure which also shows the gravitational acceleration g pointing down.



(a) Write down the Lagrangian for this system using the angle θ as the generalized coordinate. As a result of the motion of the point of suspension, the Lagrangian will be explicitly time-dependent while the Euler-Lagrange equations still hold.

(b) Write down the Euler-Lagrange equation of motion for θ and determine its equilibrium value θ_0.

(c) Determine the frequency of small oscillations about the equilibrium point θ_0 as a function of g and a. What happens in the limit $a \to 0$?

Problem 2

Consider the one-dimensional motion of a quantum mechanical particle of mass m in the potential (with $V_0 > 0$),

$$V(x) = \begin{cases} 0 \ \text{ for } x < -a \ \text{ (region I)} \\ -V_0 \ \text{ for } -a < x < 0 \ \text{ (region II)} \\ +\infty \ \text{ for } x > 0 \ \text{ (region III)} \end{cases}$$

Suppose a beam of particles of energy $E = \hbar^2 k^2 / 2m$ is incident on this potential from $x = -\infty$.

(a) Write down the wave functions in the three regions listed above and clearly state the continuity conditions that hold at the region boundaries.

(b) Obtain an expression for the coefficient of the reflected wave relative to that of the incident one in region I. What is the magnitude of this ratio?

Problem 3

A standing electromagnetic wave with electric field $\boldsymbol{E}(x,t) = \boldsymbol{E}_0 \cos kx \ \cdot \cos \omega t$ is sustained in vacuum in a source-free region along the x-axis, where $\boldsymbol{E}_0$ is a constant vector.

(a) What can you say about the components of $\boldsymbol{E_0}$? Find the corresponding magnetic field $\boldsymbol{B}(x,t)$.
(b) Calculate the Poynting vector $\boldsymbol{S}$ and the energy density u for this electromagnetic wave and verify that they satisfy the local energy conservation equation.
(c) Find the time averages of the Poynting vector and energy density over one oscillation period.

Problem 4

A spaceship (SS) moves away from Earth (E) at a speed $v = \beta c$ and releases a shuttle craft (SH) in the forward direction (say, the x-direction) at a speed v relative to the spaceship. In turn, the pilot of the shuttle craft launches a probe (PR), again in the same forward direction, at a speed v relative to the shuttle craft.

(a) Determine the speed v_{SH}(E) of the shuttle craft relative to the Earth.
(b) Determine the speed v_{PR}(E) of the probe relative to the Earth
(c) What do you expect the speed of the probe relative to the Earth to be in the limits $\beta \ll 1$ and $\beta \to 1$? Verify that the result you found in part (b) correctly reproduces your expectation in both of these limiting cases.

Problem 5

The possible states of a system are distributed in energy according to the so-called "Hagedorn spectrum", with $E \geq 0$,

$$\frac{dn}{dE} = \alpha\, E^3 \exp\,(\beta_0 E)$$

where $\beta_0 = 1/E_0$ and E_0 is some energy scale which is often expressed in terms of the "Hagedorn temperature" $T_0 = E_0/k$. This type of exponential spectrum may describe the strongly interacting particles known as mesons and baryons, and also is a feature of string theory. Evaluate, for temperatures $T < T_0$ (i.e. $\beta > \beta_0$) :

(a) The partition function Z, and

(b) The average energy, U, and the specific heat, C, of this system, written in terms of T and T_0.