LONG PROBLEMS

1: Consider the system shown in Figure 1: Two objects, of mass m_1 and m_2, can be treated as point-like. Each of them is suspended from the ceiling by a wire of negligible mass, and of length L. The two objects are connected to each other by a spring, of spring constant k. The spring is relaxed when the two wires are vertical, as shown in the Figure. Denote by θ_1 and θ_2 the angles that the two wires form with the vertical, for an arbitrary position of the two masses ($\theta_1 = \theta_2 = 0$ in the figure). Consider small oscillations of the two masses in the plane of the Figure, about the equilibrium position shown in the Figure.

(i) Find the angular frequencies ω of the two normal modes of the system (in other terms, the eigenfrequencies of the system). (ii) Provide the physical explanation of why the eigenfrequencies are what they are. (iii) Find the time evolution $\theta_1(t)$ and $\theta_2(t)$ for small oscillations of the two objects, starting at rest from the initial values $\theta_1 = 0$, and $\theta_2 = \epsilon$.

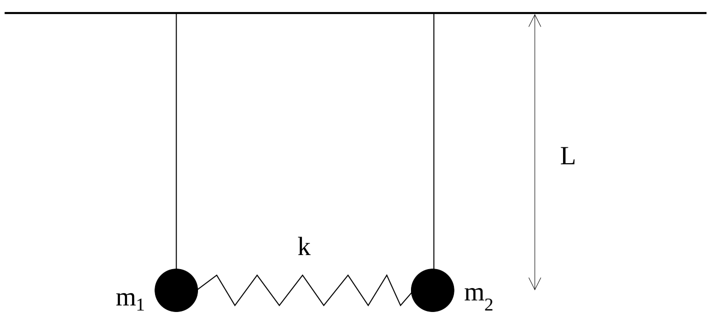

Figure 1: Long problem 1

2: The hamiltonian for a rigid body is

$$H = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right) \tag{1}$$

where I_i are the principal moments of inertia and $(L_1, L_2, L_3) = (L_x, L_y, L_z)$ are the angular momentum operators. This Hamiltonian describes the rotational spectrum of molecules.

(i) First consider the case $I_1 = I_2 = I_3 = I$ - that is, a spherical top, such as methane. Write down a formula for the energy levels in terms of an appropriate quantum number.

(ii) Next consider the case $I_1 = I_2 = I_0 \neq I_3$ - that is, a symmetric top, such as ammonia. Show that $[L_3, H] = 0$ and $[L^2, H] = 0$, and write the Hamiltonian in terms of L_3 and L^2. Then write down a formula for the energy levels in terms of appropriate quantum numbers. Indicate the allowed ranges of the quantum numbers and indicate if there are any energy degeneracies.

(iii) Now consider a slightly asymetric top, such as water molecule, for which we can write

$$\begin{aligned} I_1 &= I_0 - \alpha \\ I_2 &= I_0 + \alpha \end{aligned} \tag{2}$$

where α is small. Write down the Hamiltonian for this system as a sum of the symmetric top Hamiltonian (from part (ii)) and a perturbation term. Find the shifts in energy levels relative to those of the symmetric top, using perturbation theory to first order in α. Be complete and list the energy shifts for all values of the two quantum numbers in part (ii).

3: A nonrelativistic particle of mass m moves in one dimension in a square well potential with walls of infinite height at a distance L apart (to be concrete, you may take $V = 0$ for $0 < x < L$ and $V = +\infty$ elsewhere). At time $t = 0$ the particle is in a state with equal admixtures (with zero relative phase) of the two lowest energy eigenstates of the system.

(i) Write down the wavefunction for this particle at $t = 0$ and at an arbitrary later time t.

(ii) Calculate and plot as a function of time the probability that the particle will be found in the right-hand side of the well, i.e. at some $x > L/2$.

4: Consider a gas of (nonrelativistic) spin 1/2 fermions, of mass m, in the volume V. Denote by $n_\pm$ the densities of the fermions with spin up and spin down, respectively.

(i) Assuming that the ground state for this system can be described as two Fermi spheres, one for spin up and the other for spin down, find the ground state kinetic energy in terms of n_+ and n_-.

(ii) Assuming now that the average densities deviate slightly from average, $n_\pm \approx (n/2)\,(1 \pm \delta)$, expand the kinetic energy to the lowest nontrivial order in δ.

(iii) Assume that the interactions between these fermions can be described in terms of a potential energy term U of the form:

$$\frac{U}{V} = \alpha\, n_+\, n_- \tag{3}$$

Add the potential energy to the kinetic energy obtained above, and find the total energy to lowest nontrivial order in δ. Show that for small α one has $\delta = 0$ in the ground state, but that for $\alpha > \alpha_c$ where α_c is a critical value that you should find as a function of n, the ground state is ferromagnetic (that is, $\delta \neq 0$).

5: A solenoid, of radius a and length $L \gg a$, has n turns per unit length of wire carrying the current I. An insulating cylindrical shell with negligible thickness, radius $b > a$, and length $L \gg b$, is placed with the axis of the shell coinciding with the axis of the solenoid. The shell has a total mass M and a total charge Q, distributed uniformly on it. The current through the solenoid is reduced from $I = I_0$ to $I = 0$ over some time interval.

(i) Discuss (very briefly) what happens to the shell during this time interval. Compute the velocity acquired by the shell after the current of the solenoid has been dropped to zero. Ignore any effects of the magnetic field produced by the shell (assume that the shell can move without any mechanical friction).

(ii) Answer the same questions in part (i), including the magnetic field induced by the shell. Discuss under which condition the magnetic field produced by the shell can be ignored.

(iii) Derive a relation between the initial vector potential of the magnetic field of the solenoid at the radius of the shell and the final velocity of the shell (the approximated one obtained in part (i)). Notice that the vector potential is not uniquely defined from the magnetic field; however, choose the vector potential for which the relation to be derived is as simple as possible.

SHORT PROBLEMS

1: A positively charged particle Σ^+ decays into a neutron n and a pion π^+. Both the neutron and pion are observed to move in the same direction as the Σ^+ particle was originally moving, with momentum $p_n = 4702\,\mathrm{MeV}/c$, and $p_{\pi^+} = 171\,\mathrm{MeV}/c$. What is the mass of Σ^+? The rest energy of the neutron and the charged pion are, respectively, $E_{0n} = 940\,\mathrm{MeV}$, and $E_{0\pi^+} = 140\,\mathrm{MeV}$.

2: The spin-orbit interaction for an electron in a hydrogen atom is governed by the Hamiltonian

$$H_{\mathrm{SO}} = \frac{1}{4\pi\epsilon_0}\,\frac{e^2}{2\,m^2\,c^2\,r^3}\,\vec{L}\cdot\vec{S} \tag{4}$$

where ϵ_0 is the vacuum permittivity, e and m are the electron's electric charge and mass, respectively, c is the speed of light, r is the radial distance of the electron from the proton, and $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momentum of the electron.

(i) Compute the energy correction due to the spin-orbit term. You may need the following identity:

$$\langle\frac{1}{r^3}\rangle = \frac{2}{a^3\,n^3\,l\,(l+1)\,(2l+1)} \tag{5}$$

where a is the Bohr radius and n and l are the principal and orbital quantum numbers, respectively (notice the identity holds for $l \neq 0$)

(ii) Describe the energy level splitting for $n = 2$ (use spectroscopic notation).

3: Consider the positronium, namely a bound state of a positron and an electron. What is the corresponding Bohr radius ? What is the energy corresponding to the $(n = 2) \to (n = 1)$ transition ?

4: Two conductors of arbitrary shape are placed (without touching each other) in a liquid with a uniform conductivity σ. At $t = 0$ a total charge of $+Q_0$ is placed on one of the conductors, and $-Q_0$ on the other. Derive the time dependence of the charge on the conductors as a function of time.

5: Wind-driven currents in a body of water on the Earth spiral down due to a combination of viscous and Coriolis forces. The pitch of this spiral (called Ekman spiral in oceanography) is a length λ which depends on the water density ρ, the viscosity η, and the angular speed of the Earth rotation ω. Assuming that $\lambda \propto \rho^a \eta^b \omega^c$, use dimensional analysis to find the exponents a, b, and c.

6: A container of volume V is divided into two parts by a sliding partition. On one side there is one mole of an ideal gas made of spin 1/2 particles, and on the other side one mole of an ideal gas consisting of spin zero particles. The two gases have the same temperature T. Find the ratio of the volumes occupied by the two gases (i) at $T = 0$ and (ii) at very high T.

7: A cylindrical container of total volume $4V$, thermally insulated, is separated into two compartments of volumes V and $3V$ by a non-insulated partition. The partition is initially fixed: the smaller compartment holds one mole of an ideal gas, and the larger one six moles of a different ideal gas. The system has the temperature T_0. The partition is then allowed to slide inside the container (see Figure 2) until the system reaches equilibrium.

(i) What is the final temperature?
(ii) What are the final volumes?
(iii) What is the change in entropy in this process?

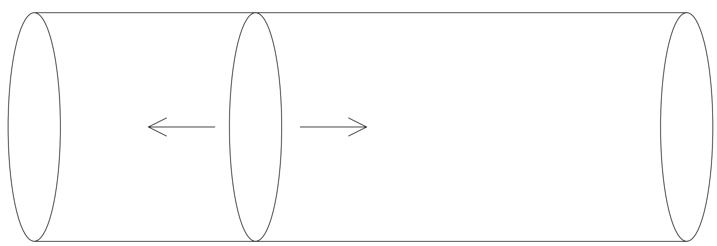

Figure 2: Short problem 7

8: Consider a monovalent simple cubic metal in which the interactions between the electrons and the lattice are so weak that the electrons can be treated as free. (i) Calculate the Fermi wavevector k_F in terms of the lattice spacing a. (ii) Show that the minimum energy that a photon must have to be absorbed by this metal is (approximately) 0.063 times the Fermi energy. Hint: You may use free electron bands in the reduced zone scheme.

9: An astronaut is on the surface of a spherical asteroid, with a uniform density equal to the average density of Earth. Estimate the condition on the radius of the asteroid, for which the astronaut can escape from the asteroid with a jump (give a numerical answer).

10: A yoyo consists of two disks of radius R_2 connected by an axle of radius $R_1 < R_2$ (see Figure 3). The yoyo descends under the influence of gravity by the unwinding of a string wrapped around its axle (the top end of the string is kept fixed). The total mass of the yoyo is m, and the axle and the string have negligible mass. Compute the downward acceleration of the yoyo's center of mass, and the tension in the string.

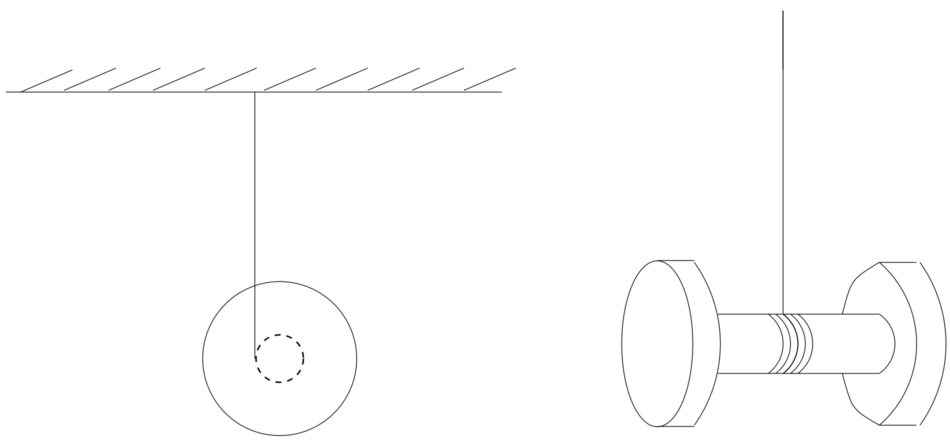

Figure 3: Short problem 10