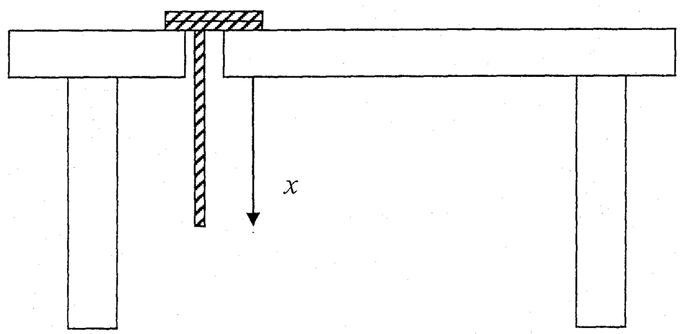
Short Problems

1. A smooth rope of length L and mass m is placed above a hole in a table. One end of the rope falls through the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity of the rope as a function of the distance to the end of the rope, x. Ignore friction of the rope as it unwinds. Then find the acceleration of the falling rope and the mechanical energy lost from the rope as the end of the rope leaves the table. Note that the rope length is less than the height of the table.



2. If an impulse is delivered to the end of a uniform rod of length l, lying on a frictionless plane, how far will it travel while making one revolution? The impulse is in the plane of the table and perpendicular to the rod.

3. A time-independent magnetic field is given by $\mathbf{B} = 2xyb\hat{\mathbf{i}} + y^2a\hat{\mathbf{j}}$.

   (a) What is the relationship between the constants a and b?

   (b) Determine the steady current density $\mathbf{J}$ that gives rise to this field.

4. A set of four point charges, q_1, q_2, q_3, and q_4, are arranged collinearly along the z-axis at $z_1 = 0$, $z_2 = a$, $z_3 = 2a$, and $z_4 = 4a$, respectively and the resulting electric field at a distant point $\mathbf{r}$ ($r \gg a$) decays *faster* than $1/r^3$. Determine the values of q_1 and q_4 when $q_2 = +2$ and $q_3 = +4$. Units for all charges are Coulombs.

5. The Lyman-α transition in atomic hydrogen has a wavelength $\lambda = 121.5$ nm, and a transition rate of 0.6×10^9 sec^{-1}. Estimate the minimum value of $\Delta\lambda/\lambda$.

6. Let n be the number of molecules in one cm^3 of air in the room you are sitting in right now. Estimate (in order of magnitude) the standard deviation of n from its average value.

7. The electron in a hydrogen atom is in a state described by the following superposition of normalized energy eigenstates u,

$$\Psi(r, \theta, \phi) = [3u_{100} + Au_{211} - 2u_{21-1} + 3u_{321}]/5$$

where the subscripts represent the quantum numbers $\{n, l, m_l\}$.

(a) Calculate $A > 0$ such that the wave function is normalized.

(b) Find the expectation value of the energy in this state, in terms of the ground state energy of hydrogen E_1.

(c) Find the expectation values of $\mathbf{L^2}$ and L_z in this state.

8. Two moles of argon (considered an ideal gas) are expanded in a process that doubles both its volume and pressure. Find the amount by which the entropy of the gas changes as result of this process in J/K.

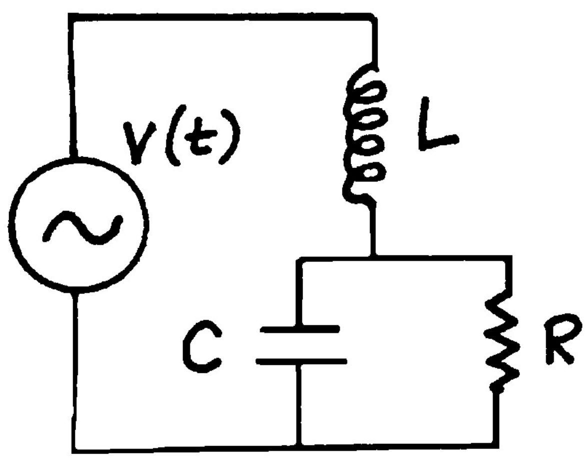
9. The magnetic moment per atom in a solid has magnitude $\mu = 10^{-23}$ J/T. What magnetic field, in tesla, must be applied at $T = 77$ K if twice as many atoms are to have their magnetic moments aligned parallel to the field as there are antiparallel?

10. A pion at rest in the lab frame decays into a muon and a neutrino ($\pi \to \mu + \nu$). Assuming that the neutrino mass can be neglected in comparison to the masses, m_π and m_μ, of the pion and muon, show that the speed v of the muon in the lab frame is given by

$$\beta = \frac{v}{c} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

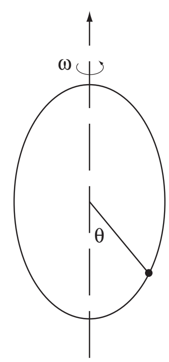
11. A rock is found to contain 4.20 mg of ^{238}U and 2.00 mg of ^{206}Pb. Assume that the rock contained no lead at the time of its formation, so that all the ^{206}Pb now present is due to the decay of the uranium originally present in the rock. Find the age of the rock given that the half-life of ^{238}U is 4.47 $\times 10^9$ yr. The decay times of all intermediate elements are negligibly short and ignore any differences in the binding energies.

12. The applied AC voltage in the circuit is given by $V(t) = V_0 \sin \omega t$, with a frequency fixed at $\omega = 1/(LC)^{1/2}$. Determine the steady state amplitude and phase of the current through the resistor R. Express your answer in terms of the amplitude V_0 of the applied voltage and the other circuit parameters.



Long Problems

1. A particle of mass m, is constrained to move without friction on a circular wire of radius R rotating with constant angular frequency ω about a vertical diameter. Gravity can not be neglected.



(a) Write down the Lagrangian for the system and the equations of motion.

(b) Find the equilibrium position(s) of the particle and determine whether this position is stable.

(c) Calculate the frequency of small oscillations about any stable points.

2. The general solution of the Laplace's equation for an electro-static problem having azimuthal symmetry can be written as

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l/r^{l+1})P_l(\cos\theta).$$

Now consider the following problem. A solid spherical conductor of radius R having charge Q is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$.

(a) Qualitatively describe the electric field inside and outside of the sphere.

(b) Solve the problem and find the electric potential in the region outside the sphere.

3. Consider the transmission of a beam of particles of mass m and momentum $p = \hbar k$, in one dimension, incident on a rectangular potential barrier of height V_0 and extending from $x = 0$ to $x = L$, in the special case that the energy E of the incident particles *is exactly equal* to the barrier height V_0.

   (a) Calculate the transmission coefficients T and R.

   (b) Check some properties of your answers in (a): is probability conserved? Do T and R have the expected limiting values for L very large or very small?

   (c) For what values of the de Broglie wavelength of the particles is the transmitted fraction equal to 1/2?

4. Consider a one-dimensional infinite array of points labeled by an index n and separated by a fixed unit distance. At each point there is an identical very deep and narrow potential well. Let $|n\rangle$ denote an eigenstate of a *single* well, with energy E.

   (a) Argue that if the wells are so narrow that the different sites can be considered uncoupled, then $|n\rangle$ is an eigenstate of the total Hamiltonian, H, with eigenvalue, E. What is its degeneracy? Then show that the state $|k\rangle$ defined as:

$$|k\rangle = \sum_{n=-\infty}^{\infty} e^{ink}|n\rangle$$

   with $-\pi < k < \pi$, is an eigenstate of both H and of the translation operator, T defined as $T|n\rangle = |n+1\rangle$. Find the respective eigenvalues.

   (b) Assume now that neighboring sites are weakly coupled so that the total Hamiltonian can now be written as

$$H = \sum_{n=-\infty}^{\infty} [|n\rangle E\langle n| - |n+1\rangle D\langle n| - |n\rangle D\langle n+1|]$$

   where the coupling parameter D is real and we assume that $\langle n|n'\rangle = \delta_{n,n'}$. Show that $|n\rangle$ is no longer an eigenstate of H but that $|k\rangle$ still is. Find the eigenvalue.

5. Two monatomic ideal gases, each occupying a volume $V = 1.0$ m^3, are separated by a removeable insulating partition. They have different temperatures $T_1 = 350$ K and $T_2 = 450$ K, and different pressures $p_1 = 10^3$ N/m^2 and $p_2 = 5 \times 10^3$ N/m^2. The partition is removed, and the gases are allowed to mix while remaining thermally isolated from the outside.

   (a) What are the final temperature T_f (in K) and pressure p_f (in N/m^2)?

   (b) What is the net change in entropy due to mixing (in J/K)?

6. A rocket passes Earth at a speed $v = 0.6c$. When a clock on the rocket says that one hour has elapsed since passing, the rocket sends a light signal back to Earth.

   (a) Suppose that the Earth and rocket clocks were synchronized to zero at the time of passing. According to the *Earth* clocks, when was the signal sent?

   (b) According to the *Earth* clocks, when did the signal arrive back on Earth?

   (c) According to the *rocket* clocks, how long after the rocket passed did the signal arrive back on Earth?